TDC part I (1) paper 1 Group A

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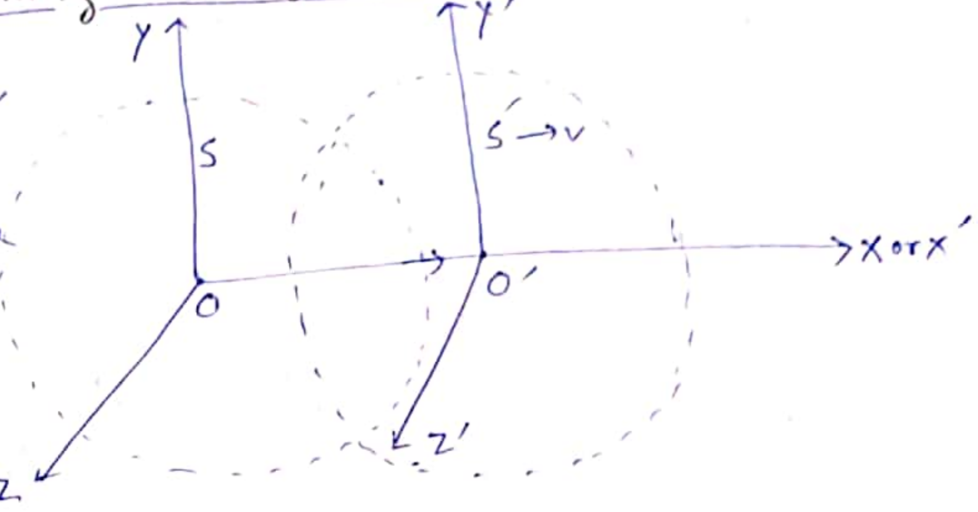
Fundamental postulates of special theory of relativity

There are two fundamental postulates of special theory of relativity given by Einstein.

1. principle of equivalence :- Laws of physics are remain same in all inertial frames of refrence. No prefered inertial frame exists.
2. principle of the constancy of velocity of light :- The speed (velocity) of light remains equal in all inertial frames of refrence.

Derivation of Lorentz Transformation Equations.

Suppose S and S′ are two inertial frames of refrence and the frame S′ is moving with velocity v along +ve x-axis relative to the frame S.



Suppose x axes of the two system are always coincident and velocity v is parallel to x axis.

Suppose when the two origins O and O′ coincide then the times on the two clocks in each frame of refrence is taken to be zero i.e, $t = t' = 0$.

Now consider an event occurs at the point (x, y, z, t) as measured in frame S and the same event occurs at the point (x', y', z', t') as measured in the frame S′.

Here relative motion of the two refrence frames S and S′ is in the x direction so all distances measured at right angle to the x direction will be the same in both refrence frames S and S'.

i.e, $y' = y$ and $z' = z$ ——— (1)

Let $x' = A(x - vt)$ ——— (2)

$t' = Bx + Dt$ ——— (3)

Where A, B and D are constants.

Suppose spherical wave front is originated from origin at $t = t' = 0$. The observers in both frames S and S' will find that the spherical wave front is emerging out from their respective centres (origins) with the same speed (velocity) c of light.

Hence equation of the spherical wave front produced from the origin O in the frame S at any time 't' is given by

$$x^2 + y^2 + z^2 = c^2t^2 \quad \text{——— (4)}$$

and the equation of the spherical wave front produced from the origin O' in the frame S' at any time 't'' is given by

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \quad \text{——— (5)}$$

c is c primed (c') because velocity of light c remains same in all inertial frame.

Now using equ^ns (1), (2), and (3) in equ^n (5), we get

$$A^2(x - vt)^2 + y^2 + z^2 = c^2(Bx + Dt)^2$$

$$\Rightarrow A^2(x^2 + v^2t^2 - 2xvt) + y^2 + z^2 = c^2(B^2x^2 + D^2t^2 + 2BxDt)$$

$$\Rightarrow (A^2 - B^2c^2)x^2 + y^2 + z^2 - 2xt(A^2v + BDc^2) = (c^2D^2 - A^2v^2)t^2 \quad \text{——— (6)}$$

Comparing equ^n (6) with equ^n (4), we get

$A^2 - B^2c^2 = 1$ ——— (7), $A^2v + BDc^2 = 0$ ——— (8), $c^2D^2 - A^2v^2 = c^2$ ——— (9)

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Equⁿ (7) × V − equⁿ (8), We get

$$A^2V - B^2c^2V - A^2V - BDc^2 = V \Rightarrow -Bc^2(BV + D) = V \quad \text{—— (10)}$$

Equⁿ (8) × V + Equⁿ (9), We get

$$A^2V^2 + BDc^2V + c^2D^2 - A^2V^2 = c^2 \Rightarrow c^2D(BV + D) = c^2$$

$$\Rightarrow D(BV + D) = 1 \quad \text{—— (11)}$$

Equⁿ (10) ÷ Equⁿ (11) $\Rightarrow -\frac{Bc^2}{D} = V \Rightarrow B = -\frac{V}{c^2}\cdot D$ —— (12) put in equⁿ (11)

$$D\left(-\frac{V}{c^2}D\cdot V + D\right) = 1 \Rightarrow D^2\left(1 - \frac{V^2}{c^2}\right) = 1$$

$$\Rightarrow D^2 = \frac{1}{1 - \frac{V^2}{c^2}} \Rightarrow D = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \quad \text{—— (13)}$$

put in equⁿ (12)

We get $B = -\frac{V}{c^2}\cdot\frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$ —— (14) put in equⁿ (7),

We get $A^2 - \frac{V^2}{c^4}\cdot\frac{1}{1 - \frac{V^2}{c^2}}\cdot c^2 = 1 \Rightarrow$ ~~$A^2 = 1 + \frac{V^2/c^2}{\ldots}$~~

$$\Rightarrow A^2 = 1 + \frac{V^2/c^2}{1 - V^2/c^2} = \frac{1 - V^2/c^2 + V^2/c^2}{1 - \frac{V^2}{c^2}} \Rightarrow A^2 = \frac{1}{1 - \frac{V^2}{c^2}}$$

$$\Rightarrow A = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \quad \text{—— (15)}$$

put in equⁿ (2), we get

$$x' = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\cdot(x - Vt) \Rightarrow x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} \quad \text{—— (16)}$$

putting values of D and B from equⁿˢ (13) and (14) in equⁿ (3)

We get

$$t' = -\frac{V}{c^2}\cdot\frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}x + \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\cdot t$$

$$\Rightarrow t' = \frac{t - \frac{V}{c^2}\cdot x}{\sqrt{1 - \frac{V^2}{c^2}}} \quad \text{—— (17)}$$

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Thus Lorentz transformation equations are

$$\boxed{x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z \text{ and } t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}}}$$

Using abbreviation, $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, the Lorentz transformation equations will be

$$\boxed{x' = \gamma(x - vt), \quad y' = y, \quad z' = z \text{ and } t' = \gamma\left(t - \frac{v}{c^2}x\right)}$$

Case 1 :- In the classical limit, i.e. $v \ll c$, the Lorentz transformation reduces to Galilean transformation.

For $v \ll c$, $\frac{v^2}{c^2} \approx 0 \Rightarrow \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0^2}} = 1$, $\frac{vx}{c^2} \approx 0$

In this case, Lorentz transformation equation will become

$$\boxed{x' = x - vt, \quad y' = y, \quad z' = z \text{ and } t' = t}$$

Which is Galilean transformation.

Note :- Lorentz transformation is used for relativistic case i.e, for $v \approx c$ case.

But Galilean transformation is used for non relativistic case i.e, for $v \ll c$ case.

Case II :- If the refrence frame S is seen from the frame S′ then it will appear that the frame S will be moving with velocity $-v$ along $-ve$ x axis relative to the frame S′.

Thus Inverse Lorentz transformation equation will be

$$x = \gamma(x' + vt'), \quad y = y', \quad z = z' \text{ and } t = \gamma\left(t' + \frac{v}{c^2}x'\right).$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

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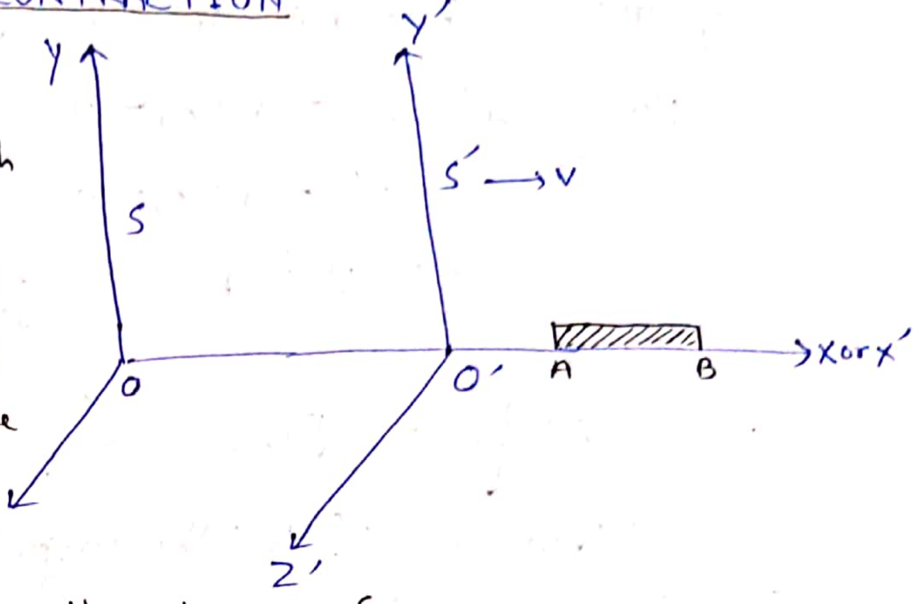
LENGTH CONTRACTION

Suppose S' be an inertial frame in which a rod AB is at rest. i.e, the frame S' is proper frame. The ~~rod~~ frame S' as well as the rod AB are moving with velocity v along +ve x-axis relative to the frame S.



Events:

E_1 : observer in the frame S, measures the coordinate of end A of the rod AB as (x_1, t_1).

E_2 : observer in the frame S, measures the coordinate of end B of the rod AB as (x_2, t_2).

$x_2 - x_1 = L$ = length of the rod AB observed in frame S only if $t_1 = t_2 = t$ (say).

From Lorentz transformation,

$$x_1' = \gamma(x_1 - vt) \quad , \quad t_1' = \gamma\left(t - \frac{v}{c^2}x_1\right)$$

and $$x_2' = \gamma(x_2 - vt) \quad , \quad t_2' = \gamma\left(t - \frac{v}{c^2}x_2\right)$$

Here t_1 and t_2 are same as t but $x_1 \neq x_2$ in the frame S so $t_1' \neq t_2'$ in the frame S'

still $x_2' - x_1' = L'$ = length of the rod AB in the frame S' because the rod is at rest in the frame S'.

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Measurement time has to be adjusted same in the frame S but not in the frame S'.

Now $x_2' - x_1' = \gamma(x_2 - vt) - \gamma(x_1 - vt)$

$$L' = \gamma(x_2 - x_1)$$

$$\Rightarrow L' = \gamma \cdot L = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot L$$

$$\Rightarrow \boxed{L = \sqrt{1-\frac{v^2}{c^2}} \cdot L'} \text{ —— (A)}$$

Thus $L < L' \Rightarrow$ The length is contracted.

Equⁿ (A) shows that the rod placed in the moving frame S', appears to be contracted by a factor of $\sqrt{1-\frac{v^2}{c^2}}$ to the stationary observer in the frame S.

This contraction occurs in the direction of their relative motion.

* If the relative velocity v between the two frame is parallel to x axis then y and z dimensions of the rod remains same in the both frames S and S'.
* If $v = c$ then $L = 0$. It means that a rod moving with velocity of light ($v=c$) will appear to reduce to a point object to a stationary observer.

Note : * proper frame :- The refrence frame in which the rod is kept stationary, is known as proper frame.

* proper length :- proper length of a rod is the length measured in the frame in which the rod is at rest.
* Its proper length is largest. Its length is shorter in other inertial frame.

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TIME DIALATION

proper time interval :- The time interval between two events measured in an inertial frame in which the two events occur at the same position, is known as proper time interval.

Suppose two events occur at the same position x' in the inertial frame s' at times t_1' and t_2' as measured in the frame s'.

proper time interval $= t_1' - t_2' = t'$.

The inertial frame s' is moving with velocity v along +ve x axis relative to the inertial frame s. If t_1 and t_2 be times measured in the frame s for the two events then

time interval between the two events is $t = t_2 - t_1$.

From inverse Lorentz transformation,

$t_1 = \gamma\left(t_1' + \frac{v}{c^2}x'\right)$ and $t_2 = \gamma\left(t_2' + \frac{v}{c^2}x'\right)$

Now time interval $t = t_2 - t_1$

$$\Rightarrow t = \gamma\left(t_2' + \frac{v}{c^2}x'\right) - \gamma\left(t_1' + \frac{v}{c^2}x'\right)$$

$$\Rightarrow t = \gamma\left(t_2' - t_1'\right)$$

$$\Rightarrow t = \gamma \cdot t' \Rightarrow \boxed{t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot t'} \Rightarrow t > t'$$

B

Therefor, the time interval t is dialated by a factor of $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ in comparision with proper time interval t'. i.e, a stationary observer in the frame S measures longer time interval between the two events occuring in the frame S'. In other words, a moving clock with frame S' appears to be slowed down to a stationary observer in the frame S. This phenomenon is known as time dialation.

* If $v = c$ then $t = \infty$, It means that a clock moving with the frame S' moving with velocity equal to Velocity of light will appear to be completely stopped to a stationary observer in the frame S.

Ques:- State the fundamental postulates of special theory of relativity and derive Lorentz transformation equation.

Ques:- Explain length contraction and time dialation.

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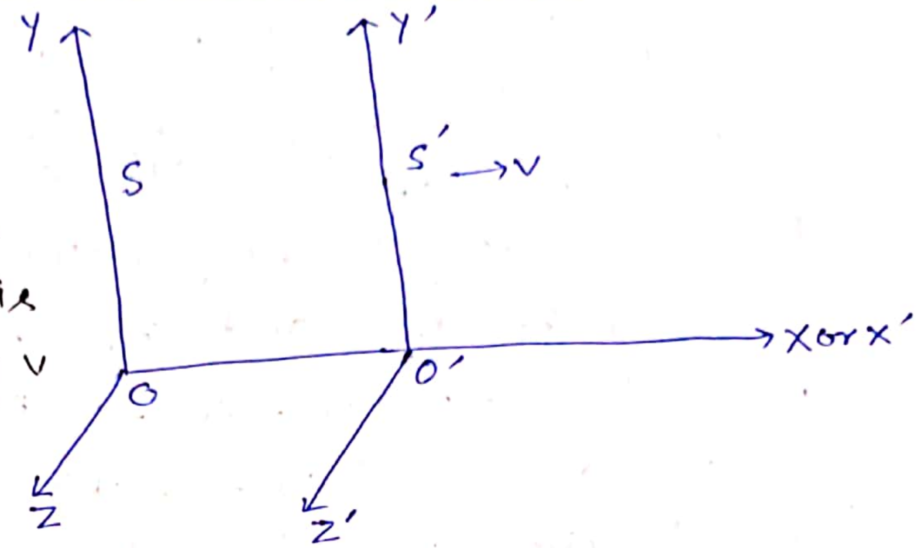
VELOCITY TRANSFORMATION EQUATION

Suppose S and S′ are two inertial frames of refrence and the frame S′ is moving with velocity v along +ve x-axis relative to the frame S.



Suppose a particle moving neither in frame S nor in frame S′ is observed by the observers attached with the frames S and S′.

Instantaneous velocity of the particle at any instant is observed as u and u′ in the frames S and S′ respectively.

The equation relating u and u′ is known as velocity transformation equation.

Imagine that a particle is moving along x-direction observed in the frame S.

E_1: particle has coordinate x_1 at time t_1.

E_2: particle has coordinate x_2 at time t_2.

Even if the velocity of the particle is not constant,

$$\frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

Instantaneous velocity of the particle in the frame S along x-axis is given by

$$u_x = \mathop{Lt}_{\Delta t \to} \frac{\Delta x}{\Delta t}$$

If motion of the particle be in three dimensional then

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In general,

$$u_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}, \quad u_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \text{ and } u_z = \lim_{\Delta t \to 0} \frac{\Delta z}{dt}.$$

Similarly, if the motion of the particle is observed in the frame S' then

$$u'_x = \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'}, \quad u'_y = \lim_{\Delta t' \to 0} \frac{\Delta y'}{\Delta t'} \text{ and } u'_z = \lim_{\Delta t' \to 0} \frac{\Delta z'}{\Delta t'}$$

From Lorentz transformation,

$$x'_1 = \gamma(x_1 - Vt_1) \quad \text{and} \quad x'_2 = \gamma(x_2 - Vt_2)$$

Now $x'_2 - x'_1 = \gamma\left[(x_2 - x_1) - v(t_2 - t_1)\right]$

$$\Rightarrow \Delta x' = \gamma\left[\Delta x - V\Delta t\right]$$

Thus Lorentz transformation can be written in ~~differ~~ differential form as

$$\Delta x' = \gamma(\Delta x - V\Delta t), \quad \Delta y' = \Delta y, \quad \Delta z' = \Delta z \text{ and } \Delta t' = \gamma\left(\Delta t - \frac{V}{c^2}\Delta x\right)$$

Now $\dfrac{\Delta x'}{\Delta t'} = \dfrac{\gamma(\Delta x - V\Delta t)}{\gamma\left(\Delta t - \frac{V}{c^2}\Delta x\right)} = \dfrac{\frac{\Delta x}{\Delta t} - V}{1 - \frac{V}{c^2}\cdot\frac{\Delta x}{\Delta t}} \Rightarrow \boxed{u'_x = \dfrac{u_x - V}{1 - \frac{V}{c^2}\cdot u_x}}$

Again $\dfrac{\Delta y'}{\Delta t'} = \dfrac{\Delta y}{\gamma\left(\Delta t - \frac{V}{c^2}\cdot\Delta x\right)} = \dfrac{\frac{\Delta y}{\Delta t}}{\gamma\left(1 - \frac{V}{c^2}\cdot\frac{\Delta x}{\Delta t}\right)} \Rightarrow \boxed{u'_y = \dfrac{u_y}{\gamma\left(1 - \frac{V}{c^2}u_x\right)}}$

Again $\dfrac{\Delta z'}{\Delta t'} = \dfrac{\Delta z}{\gamma\left(\Delta t - \frac{V}{c^2}\Delta x\right)} = \dfrac{\frac{\Delta z}{\Delta t}}{\gamma\left(1 - \frac{V}{c^2}\cdot\frac{\Delta x}{\Delta t}\right)} \Rightarrow \boxed{u'_z = \dfrac{u_z}{\gamma\left(1 - \frac{V}{c^2}\cdot u_x\right)}}$

Thus velocity transformation equations are

$$\boxed{u'_x = \frac{u_x - V}{1 - \frac{V}{c^2}\cdot u_x}, \quad u'_y = \frac{u_y}{\gamma\left(1 - \frac{V}{c^2}u_x\right)} \text{ and } u'_z = \frac{u_z}{\gamma\left(1 - \frac{V}{c^2}\cdot u_x\right)}}$$

Inverse velocity transformation equations are

$$\boxed{u_x = \frac{u'_x + V}{1 + \frac{V}{c^2}u'_x}, \quad u_y = \frac{u'_y}{\gamma\left(1 + \frac{V}{c^2}u'_x\right)} \text{ and } u_z = \frac{u'_z}{\gamma\left(1 + \frac{V}{c^2}u'_x\right)}}$$

---x---

Q. Discuss relativistic law of addition of velocities.
or Derive velocity transformation equation
or Derive Lorentz transformation equation for velocity.

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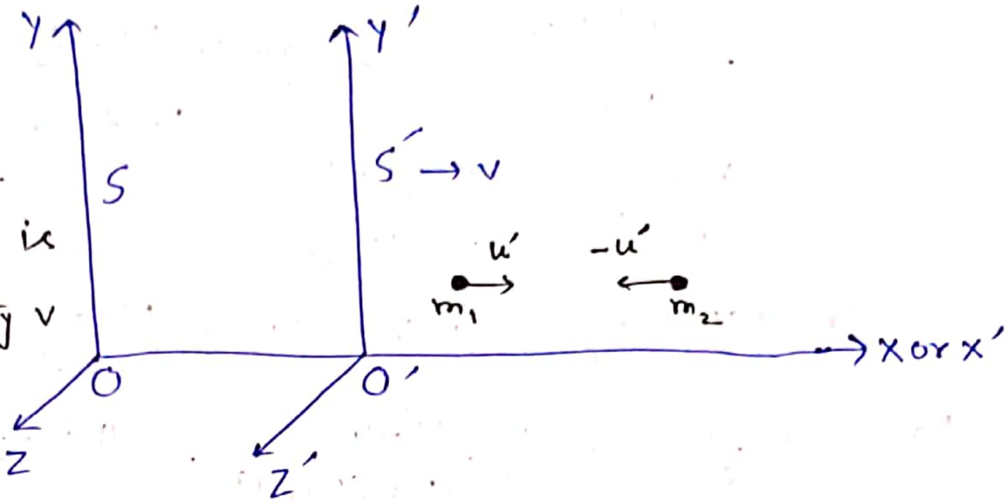
Formula for variation of mass of a particle with velocity

Suppose S and S' are two inertial frames of refrence and the frame S' is moving with velocity v along +ve X axis relative to the frame S'.



X axes of the two frames coincide.

Let us consider two particles of masses m_1 and m_2 in the frame S' which are moving with equal and opposite velocities u' and $-u'$ respectively parallel to X axis in the frame S' and they collide each other.

Suppose there is an observer in the frame S who is observing the motion and collision of the two particles. Suppose u_1 and u_2 be the velocities of the two particles respectively before collision, observed by the observer in the frame S.

Using Inverse Lorentz transformation equation of velocity,

$$u_1 = \frac{u'+v}{1+\frac{v}{c^2}\cdot u'} \quad \text{—— (1)} \quad \text{and} \quad u_2 = \frac{-u'+v}{1-\frac{v}{c^2}\cdot u'} \quad \text{—— (2)}$$

At the time of collision, the two particles will be at rest in the frame S' but they are moving with common velocity v along +ve x-axis relative to the observer in the frame S.

Using principle of conservation of momentum in frame S,

$$m_1 u_1 + m_2 u_2 = (m_1+m_2)v \quad \text{—— (3)}$$

Using eqns (1) and (2) in eqn (3), we get

$$m_1 \frac{u'+v}{1+\frac{v}{c^2}\cdot u'} + m_2 \cdot \frac{-u'+v}{1-\frac{v}{c^2}\cdot u'} = (m_1+m_2)v$$

$$\Rightarrow m_1 \left[\frac{u'+v}{1+\frac{v}{c^2}u'} - v\right] = m_2 \left[v - \frac{-u'+v}{1-\frac{v}{c^2}\cdot u'}\right]$$

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$$m_1\left[\frac{u'+v-v-\frac{v^2}{c^2}\cdot u'}{1+\frac{v}{c^2}\cdot u'}\right]=m_2\left[\frac{v-\frac{v^2}{c^2}\cdot u'+u'-v}{1-\frac{v}{c^2}\cdot u'}\right]$$

$$\Rightarrow \frac{m_1 u'\left(1-\frac{v^2}{c^2}\right)}{1+\frac{v}{c^2}\cdot u'}=\frac{m_2 u'\left(1-\frac{v^2}{c^2}u'\right)}{1-\frac{v}{c^2}\cdot u'}$$

$$\Rightarrow \frac{m_1}{m_2}=\frac{1+\frac{v}{c^2}\cdot u'}{1-\frac{v}{c^2}\cdot u'} \quad \text{———} \quad (4)$$

On squaring equ^n (1), $u_1^2=\left(\frac{u'+v}{1+\frac{v}{c^2}u'}\right)^2 \Rightarrow u_1^2=\frac{(u'+v)^2}{\left(1+\frac{v}{c^2}u'\right)^2}$

Now $1-\frac{u_1^2}{c^2}=1-\frac{1}{c^2}\frac{(u'+v)^2}{\left(1+\frac{v}{c^2}\cdot u'\right)^2}=\frac{\left(1+\frac{v}{c^2}u'\right)^2-\left(\frac{u'+v}{c}\right)^2}{\left(1+\frac{v}{c^2}u'\right)^2}$

$$\Rightarrow 1-\frac{u_1^2}{c^2}=\frac{1+\frac{v^2}{c^4}\cdot u'^2+\cancel{2\frac{v}{c^2}\cdot u'}-\frac{u'^2}{c^2}-\frac{v^2}{c^2}-\cancel{2\frac{v}{c^2}\cdot u'}}{\left(1+\frac{v}{c^2}u'\right)^2}$$

$$1-\frac{u_1^2}{c^2}=\frac{\left(1-\frac{u'^2}{c^2}\right)-\frac{v^2}{c^2}\left(1-\frac{u'^2}{c^2}\right)}{\left(1+\frac{v}{c^2}\cdot u'\right)^2}=\frac{\left(1-\frac{u'^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{v}{c^2}\cdot u'\right)^2}$$

$$\Rightarrow \left(1+\frac{v}{c^2}\cdot u'\right)^2=\frac{\left(1-\frac{u'^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{u_1^2}{c^2}} \quad \text{———} \quad (5)$$

Similarly $\left(1-\frac{v}{c^2}u'\right)^2=\frac{\left(1-\frac{u'^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{u_2^2}{c^2}}$ ——— (6)

Equ^n (5) is divided by equ^n (6), we get

$$\frac{\left(1+\frac{v}{c^2}\cdot u'\right)^2}{\left(1-\frac{v}{c^2}\cdot u'\right)^2}=\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}} \Rightarrow \frac{1+\frac{v}{c^2}\cdot u'}{1-\frac{v}{c^2}u'}=\sqrt{\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}}} \quad \text{———} \quad (7)$$

From equ^{n}s (4) and (7), we get

$$\frac{m_1}{m_2}=\sqrt{\frac{1-u_2^2/c^2}{1-u_1^2/c^2}} \Rightarrow m_1\cdot\sqrt{1-u_1^2/c^2}=m_2\cdot\sqrt{1-u_2^2/c^2} \quad \text{———} \quad (8)$$

Now if the particle of mass m_2 be at rest in the frame S then the mass m_2 is the mass of the particle at zero velocity which is usually denoted

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as m_0 (rest mass).

put $m_2 = m_0$ and $u_2 = 0$ in equⁿ (8), We get

$$m_1 \cdot \sqrt{1 - u_1^2/c^2} = m_0 \sqrt{1 - \frac{0^2}{c^2}} \Rightarrow m_1 \cdot \sqrt{1 - \frac{u_1^2}{c^2}} = m_0$$

$$\Rightarrow m_1 = \frac{m_0}{\sqrt{1 - \frac{u_1^2}{c^2}}}$$

In general, $m_1 = m$ and $u_1 = v$, we have

$$\boxed{m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}} \quad \text{——— (9)}$$

In equⁿ (9), m_0 = Rest mass of the particle (when particle at rest)
and m = kinetic mass of the particle when the particle is moving with velocity v.

From equⁿ (9), it is clear that mass of a particle varies with its velocity v. On increasing velocity of a particle, its mass increases.

* If $v << c$ then $\frac{v^2}{c^2} \approx 0$ From equⁿ (9), $m = m_0$

Thus for non relativistic case ($v<<c$) i.e, for Newtonian mechanics, kinetic mass (m) is equal to rest mass (m_0).
Therefore mass of a particle remains unchanged in nonrelativistic case. (In Newtonian Mechanics).

* If $v = c$ then from equⁿ (9), $m = \frac{m_0}{\sqrt{1 - \frac{c^2}{c^2}}} = \frac{m_0}{0} \Rightarrow m = \infty$

Therefore, if a particle moves with velocity equal to velocity of light then its mass becomes infinite.

* If $v > c$ then from equⁿ (9), $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{-ve}}$ = Imaginary

Therefore, if a particle moves with velocity greater than velocity of light then its mass becomes imaginary which is not possible. Thus a particle can not move with velocity greater than velocity of light.

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* If $m = 2m_0$ then $v = ?$

From eqn (9), $m = m_0 / \sqrt{1 - \frac{v^2}{c^2}}$

$$\Rightarrow 2m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow 2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

On squaring, $4 = \frac{1}{1-\frac{v^2}{c^2}} \Rightarrow 4 - \frac{4v^2}{c^2} = 1$

$$\Rightarrow \frac{4v^2}{c^2} = 3 \Rightarrow v^2 = \frac{3}{4}c^2 \Rightarrow \boxed{v = \frac{\sqrt{3}}{2} \cdot c}$$

Therefore, if a particle moves with velocity equal to $\frac{\sqrt{3}}{2}c$ then its kinetic mass will become double of its rest mass.

Ques:- Derive the formula for the variation of mass of a particle with its velocity. Find the velocity of the particle at which the mass of the particle become double of its ~~its~~ rest mass. (2018, 2013, 2011)

Ques: Deduce an expression showing the variation of mass with velocity. Hence show that the velocity of light is the maximum velocity that can be attained by any material particle. (2015)

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Mass-Energy Equivalence or Einstein's Mass-Energy Relation : $E=mc^2$

According to Einstein's special theory of relativity, the mass m of a body moving with velocity v relative to an stationary observer, varies with velocity v as

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}} \quad \text{——— (1)}$$

where m_0 = rest mass of the body, c = velocity of light.

The variation of mass with velocity has modified our ideas about energy.

Suppose a force F is applied on a body of mass m in the direction of velocity v.

$$\text{Force } F = \frac{dp}{dt} = \frac{d(mv)}{dt} \qquad \because p = mv$$

$$\Rightarrow F = m \cdot \frac{dv}{dt} + v \cdot \frac{dm}{dt} \quad \text{——— (2)} \qquad \because m = \text{variable}$$

From work-energy theorem, work done by the force F on the body during a small displacement ds is equal to change in kinetic energy (dK) of the body.

$$\text{Thus } dK = F \cdot ds = \left(m\frac{dv}{dt} + v \cdot \frac{dm}{dt}\right) ds$$

$$\Rightarrow dK = m \cdot \frac{ds}{dt} \cdot dv + v \cdot \frac{ds}{dt} \cdot dm$$

$$\Rightarrow dK = mv\,dv + v \cancel{x} v \cdot dm \qquad \because \frac{ds}{dt} = v$$

$$\Rightarrow dK = mv\,dv + v^2 dm \quad \text{——— (3)}$$

On differentiating equ^n (1), we get

$$dm = m_0 \cdot \frac{d\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}}{d\left(1-\frac{v^2}{c^2}\right)} \times \left(0 - \frac{1}{c^2} \cdot \frac{dv^2}{dv}\right) \cdot dv$$

$$= m_0 \left(-\frac{1}{2}\right) \cdot \left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}} \cdot \left(-\frac{1}{c^2} \cdot 2v\right) \cdot dv$$

$$= \frac{m_0 v\,dv}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{v\,dv}{c^2\left(1-\frac{v^2}{c^2}\right)}$$

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$$\Rightarrow dm = \frac{mv\,dv}{c^2 - v^2} \qquad \because m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow mv\,dv = (c^2 - v^2)\,dm \text{ ———— } (4) \text{ put in equ}^n\ (3)$$

We get, $dK = (c^2 - v^2)\,dm + v^2\,dm$

$$\Rightarrow dK = c^2\,dm \text{ ———— } (5)$$

Suppose the body has mass m_0 (rest mass) when the body is at rest and mass m when it is accelerated to velocity v. During this process, kinetic energy acquired by the body is

$$K = \int dK = c^2 \int_{m_0}^{m} dm = c^2 \left[m\right]_{m_0}^{m}$$

$$\Rightarrow K = c^2 (m - m_0) \text{ ———— } (6)$$

Therefore, kinetic energy of a body is equal to the product of relativistic increase in mass $(m - m_0)$ of the body and square of velocity of light.

It can be understood that even when a body is at rest, it possesses an amount of energy equal to m_0c^2 which is known as rest mass energy E_0 of the body of rest mass m_0. Rest mass energy $E_0 = m_0c^2$

Therefore, total energy of a body is equal to sum of kinetic energy (K) and the rest mass energy (E_0).

Thus total energy of the body is

$$E = K + E_0 \Rightarrow E = c^2(m - m_0) + m_0c^2$$

$\Rightarrow \boxed{E = mc^2}$ This is very famous Einstein's mass-energy relation.

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$E = mc^2$ relation is mass-energy equivalence.

$E = mc^2$ relation shows that the mass m of a moving body is associated with an amount of energy $E = mc^2$ or a body of total energy E has associated with its inertial mass $m = \frac{E}{c^2}$.

physical Significance of mass-energy equivalence or Importance of mass-energy equivalence

* When-ever we supply any kind of energy to a body then its total energy increases and consequently its mass increases. In fact when a body is heated then its total energy increases and consequently its mass increases i.e, it becomes more massive.

* when-ever a body releases any kind of energy then its total energy decreases and consequently its mass decreases. In fact, when a body is cooled down then its total energy decreases and consequently its mass decreases. i.e, it becomes less massive.

* process of annihilation of matter :- when a positron and an electron are annihilated then after annihilation they produce two photons or radiant energy.

$$\text{i.e,}\quad \underset{\text{positron}}{e^{+}} + \underset{\text{electron}}{e^{-}} \longrightarrow \underset{\text{EMW.}}{2\gamma\ (\text{photon})}$$

Here particles are converted into photon i.e, whole mass of positron and electron is converted into energy according to mass-energy relation $E = mc^2$.

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* During the process of nuclear fission and nuclear fusion, enormous energy released according to mass energy relation.

During the process of nuclear fission or nuclear fusion, product has a little less mass than reactant i.e, some mass is lost during the process. This lost mass is converted into energy according to Einstein's mass-energy relation $E = mc^2$. That is why enormous energy is released during the process of nuclear fission or nuclear fusion reaction.

Ques:- Obtain an expression for mass-energy equivalence and discuss its physical significance. (2017,

Ques:- Obtain the expression for mass-energy relation on the basis of special theory of relativity. Discuss its physical significance. (2019

Ques:- Establish mass-energy relation and discuss some of its consequences. (2014, 2012